The Benefits of Holding Cash: A Real Options Approach

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Abstract

Companies need to decide on the optimal amounts of cash to hold. Although this problem has long been acknowledged as a major issue for corporations, new advances in the finance literature have not been fully implemented in this area. We propose here what we believe is the first modelization of a real options approach to determine the financial benefits of holding cash. We measure the benefits of holding cash if raising new capital takes time, is costly and if the firm faces the risk of having to issue underpriced securities to obtain that capital. We show that the methodology proposed leads to non-intuitive results that warrant further research in the field and should attract academics' as well practitioners' attention.

1 Introduction

It is a well known fact in both the corporate and the academic worlds that many corporations hold large amounts of cash, at a significant cost. For example, the S&P 500 Corporations held a total of $716 billion in cash and marketable securities on their balance sheets as of fiscal year 1994 (as reported in Opler and alii (1999)). The cost of holding cash or highly liquid assets is high in many cases and can be well identified. It consists to a large extent in the opportunity cost of having to invest funds over a very short horizon in highly liquid assets with an associated low return. Another significant cost, true under many tax codes, consists in the fact that these holdings may face double taxation, as interest income from liquid assets is first taxed at the corporate level and then again when the income is distributed to shareholders. Without corporate benefits to cash holdings, shareholders would favor obtaining distribution of the cash early via a tax-efficient scheme (possibly share repurchases) and investing the cash themselves.

The necessity of identifying the true benefits of cash holdings to the corporation and thus to the shareholders becomes therefore crucial. Although this has not been a favored topic in the finance literature during the last few years (while it has been a major topic of concern in the past), some new papers in the area are revealing the need for further theoretical work in the field. There is an old literature on the topic which gained respectability notably with the work of Keynes (1936). Keynes stressed that the major benefits of cash holdings can be seen as arising from two motives, the transaction cost motive (as holding cash or liquid assets allows not to have to liquidate at higher cost less liquid assets in case of need for funds) and the precautionary motive (as highly liquid assets can be used to finance new investment and activities if other sources of funding are too costly). The profusion of academic work in the 60s (Meltzer (1963), Frazer (1964), Miller and Orr (1966), Vogel and Maddala (1967)) was mostly done in continuation of this thinking and some work in operations research has tried to implement the concept directly in order to optimize cash holdings. More recently, game theory based models (such as Myers and Majluf (1984)) have given a new reason for holding cash in identifying investment inefficiencies when firms do not hold cash and are faced with possible undervaluation on the financial markets. These models have led to a pecking order of financing that reveals the need to hold cash to be able to finance growth opportunities (something close to the precautionary motive of Keynes) especially for firms facing high information asymmetries, such as high R&D or high marketing corporates (a reason not well identified before).

Recent empirical work from Opler and alii (1999) has confirmed the pertinence of the static trade-off theory (balancing benefits versus costs of cash holdings) and confirmed that the reality somewhat validates past theories, at least qualitatively. The goal of this paper is to show that recent advances in the finance field can be used to move the analysis of the benefits of cash holdings further. We trust the methodology we propose can lead to a more precise understanding of the advantages of holding cash, to a better financial measure of these benefits of cash holdings and thus to a better determination for academics and practitioners alike of what the optimal level of cash holdings should be. The precautionary motive proposed by Keynes can easily be seen as a so-called "real option" problem by researchers familiar with the real options theory (the theory of applying the financial options analogy to real investment decisions). Interestingly enough, this can be combined with the Myers & Majluf undervaluation problem in the option framework in order to determine, based on a few variables (such as risk and loss at waiting), the financial benefits of cash holdings. This approach has not been applied to the field yet and yields interesting results (including some unexpected ones) as shown underneath.

1.1 The model

In this stylized model, we analyze the decision of a firm having an investment project of a fixed size. The firm must decide at time 0 on the amount of cash it should hold. Alternatively to increasing its cash holdings it can get new capital from outside sources. Raising new capital requires some time though (one dimension of the liquidity problem). It is well known that equity offerings take long preparation (whether seasoned or IPOs) but fixed income offerings require time as well (in the US, the occurrence of the shelf registration process has decreased the time requirements of issues, possibly an interesting event to study for a test of the theory) and even setting up bank facilities do take time. Obviously the different sources of financing (equity, bonds, credit facilities, MTN and other financing choices) have very different costs. We ignore these costs in this model as these have been well studied in the literature and stress the timing issue that has not been as well treated for the time being. A full theory would combine the cost differences of the different sources of financing with their respective timing issues. We assume here that the new capital obtained from the outside financial markets is received at time T. The firm faces two risks related to the decision on whether to hold cash or to get outside financing. Firstly it might be optimal to start the project at an earlier stage than time T. The timing of the obtention of capital may thus be a binding constraint that makes it suboptimal to rely on outside financing sources. We thus determine what is the value of holding cash as far as this timing option goes. Secondly the moment when the firm raises the outside capital could be when its stock is undervalued (the classical Myers and Majluf (1984) difficulty, which arises for equity as well as for fixed income issues, although the problem is somewhat alleviated in the second case). We also express the value of holding cash when there is a risk of undervaluation for outside financing.

We proceed by separating the two problems and examining each. First we analyze the situation where there is only uncertainty about the timing of the project. In a second step we analyze the situation where the only uncertainty comes from the level of undervaluation at time T. Hence we assume that the project can only be undertaken at time T, but the firm can raise the money today and carry the surplus cash until time T. In a third step we combine the two parts and look at the situation where the firm can raise the cash today and exercise the option at any time until the moment T. In this third step, the firm faces both the timing and the undervaluation issues. The model thus gives the value of holding cash when both timing and undervaluation are at risk. We do not provide a full solution in this paper for the overall problem as American option pricing proves itself to be rather arduous (as is well known in the option pricing field), but the methodology allows for implementation of the problem with any software allowing for the pricing of American options (via numerical analysis for example). We describe a few stylized facts that follow from the model as they appear on the timing problem or on the undervaluation problem.

1.2 The value of timing

As usual in the literature we assume, in order to reach an easy application of option pricing theory, that the value v of the project follows a standard geometric Brownian motion process. The process is given by:

$$\frac{dV}{V} = \alpha dt + \sigma dw$$

where α is the drift (the rate of return on the project) and σ is the volatility of the return on the project (and dw is a Wiener process). The project can be undertaken at any time for a fixed cost K. We assume that K is constant through time, i.e. that the investment required is fixed, whenever the project is implemented. It would be possible to extend the model to have K be a deterministic function of time, for example because inflation in construction costs makes it more and more expensive to invest as we wait, or a stochastic variable, for example because there is uncertainty about what the cost of the investment will be in the future. Both these extensions are available in the current option pricing literature. Thus the NPV of the project, should the project be launched today, is of V-K. Suppose also that there is a loss at waiting that we denote d. If the firm waits for too long, other companies may undertake similar projects before and the deteriorated competitive position of the firm makes the project less valuable. Or costs may increase through time, making it interesting to invest earlier rather than later. d is thus the percentage loss per period there is to wait.

The value of waiting to invest has long been analyzed as a standard real option. The corporate can invest at some point in time for a certain amount of investment (here K) in order to obtain the value of the project in a structure that is very similar to the structure of an option whereby an investor can buy at some point in time the value of an underlying security (for example a stock) for a certain exercise price.

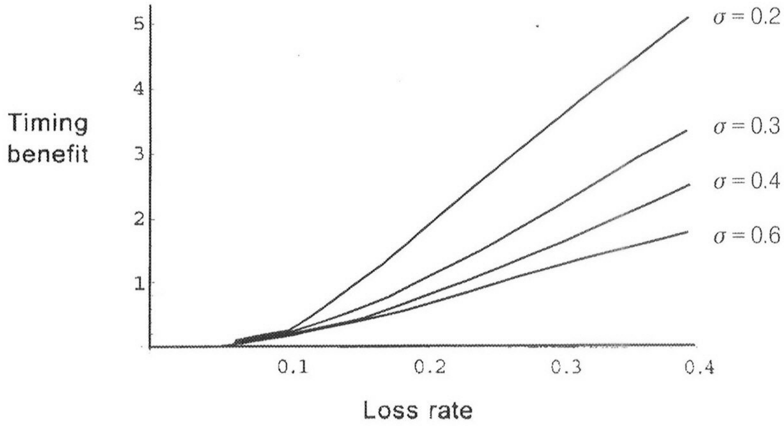
If the firm decides to get outside financing it will take some time to organize it, as discussed above. The firm will receive the funds at time T. On the other hand, if the firm has cash on hand, it can use an amount K of this cash to invest directly in the project. Therefore, if the firm is able to finance the project with cash, the value of being able to finance the project at any time corresponds to the value of an American option (i.e. a call option that can be exercised anytime before maturity) with maturity T. Indeed, the fact of having cash on hand gives rise to the possibility of investing at the cost of investment (the exercise price of K) in the value of the project: this is identical to having a call option that can be exercised at any time. If the firm chooses to raise the money from the outside the value of the project corresponds to a European option (i.e. an option that can be exercised at maturity only) with maturity at time T. We assume for simplicity that there is no interest from the firm in delaying the investment for longer than the time it needs to raise the capital from outside markets.

Hence the timing benefit from holding cash rather than having to wait to obtain the capital from outside sources is:

Timing benefit from holding cash = American Option - European Option.
The value of the European option is given by the standard Black-Scholes formula.

The value of the American option is calculated using the approximation proposed by MacMillan, which was first implemented by Barone-Adesi and Whaley (1987). The following graph shows from a computer simulation how the benefit from having the necessary cash at hand changes depending on the risk of the underlying project value and the loss occurred at waiting before investing.

Figure 1 Timing benefit from holding cash.



The above figure shows the gain from holding cash for different volatilities of the project value and different rates of loss at waiting from the project. The values of the fixed parameters are: $V_0 = 100$, $K = 90$, $r = 0.08$, $T = 0.25$. The project considered thus has an immediate NPV of 10. It is assumed that it takes a quarter (3 months) to obtain cash from the markets ($T = 0.25$).

As the graph shows, this simple analysis can already lead to intuitive and to counterintuitive results. First, the benefit from holding cash increases as the loss at waiting increases. This is an expected result. The timing gain is more important as waiting costs more. If the loss at waiting reaches a level of 20% (instantaneous continuously compounded rate, i.e. 22.14 % loss in project value at waiting one year), then the fact of having to wait for a quarter of a year leads to a loss in value on the project side of 2 (for an NPV if exercised at origination of 10 and a project return volatility of 20%). Of course, even if you have cash on hand, you may be interested in waiting before investing, something that our modelization captures as well. Overall thus we see that the timing value of having cash (or the timing loss of not having cash) can have a large impact on the value of the project.

A counterintuitive result may lie in the interpretation of the volatility impact. The above gain from holding cash is indeed divided by more than 2 if the volatility of the project return is of 60% instead of 20%. Indeed, as seen in the graph, the gain from holding cash decreases with volatility, at all levels of losses at waiting. In other words, if there is a loss at waiting, then holding cash is more interesting in a stable environment, for stable project values, than for high risk environment, high risk project values. This can seem at first in contradiction with the classical "precautionary motive" to hold cash. It is actually not, but counterbalances it some nonetheless. It is not in contradiction because the investment cost here is not stochastic, the presence of the investment itself and of the cost to start it are not subject to risk (which is the type of risk that would lead managers to think they need more cash on hand to face riskier projects). Nonetheless, it counterbalances in an interesting way the classical precautionary motive. Indeed, if investments are seen as real options, corporates having the right but not the obligation to invest and thus being able to wait before investing, then the difference between the American and the European option decreases with higher volatilities. A given project (for example a R&D project) in a risky environment (for example biotechnology) may require less cash holdings than a project in a less risky environment but with similar loss at waiting (for example a patent extension on a classical drug) just because it may optimally be more interesting to wait anyway in the first project rather than in the second one (and if we want to wait anyway, why not raise the capital from the markets?).

This simple modelization shows that incorporating the option value of investments in the understanding of the timing benefits of cash holdings first justifies possibly high level of cash holdings as the value gain, the benefits of these holdings, can be large. It also shows that one should not fully rely on the simplest intuitions as the problem, even in its simplest form, is getting quite complex and it is thus possible that higher risk projects would lead to less cash holdings (a striking stylized fact that would be mitigated by making the exercise price risky as well).

1.3 The value of avoiding the risk of undervaluation

The second risk that the firm is facing if it decides to raise the required capital from outside sources of funds (such as equity) is that it does not know ex ante the value that investors will attribute to its shares at time T. This problem was originally explored by Myers and Majluf (1984). It is possible, in their model, that a company will pass up positive NPV investments, just because it would have to issue underpriced shares to finance its projects and the underpricing may more than counterbalance the gain from the positive NPV investment. The Myers and Majluf model, while it clearly describes the problematic at hand, does not give an actual evaluation of the cash required to overcome the underpricing difficulty. This is linked to the methodology (game theory based) that they follow. We hereby provide for a solution to the problem by rederiving their argument under a real options framework and thus obtain, under some assumptions, the benefit of holding cash to avoid the risk of underpricing on the markets. We assume (as in Myers and Majluf) that the management knows the fair value of the shares at the current time. Thus management is able to see if the company is under or overvalued at any given point in time. If the company is overvalued, the management always has an incentive to invest in the project. If the company is undervalued, then management may have an incentive not to undertake a positive value investment, as long as this value does not compensate fully the loss incurred on the security issue. We assume that the amount of security undervaluation obtained at raising capital from outside markets follows a mean reverting process which is given by

$$dX = a(b - X_t)dt + \sigma_x dz$$

where a is the speed of mean reversion, b is the mean reversion level and X_t is the value of the under or overpricing at time t, σ_x is the volatility of the underpricing and dw is a standard Brownian motion. Notice that we can, from this formulation, describe situations in which the stock is on average underpriced ($b>0$) or overpriced ($b<0$) as well as situations where markets tend to revert to the true value of the underlying security but vary through time for exogenous reasons not related to the actual value of this particular stock. For example, during a security market downturn, some companies that have not seen their fundamental economic reality change with the downturn, may suddenly have to cut on their investments if they do not hold enough cash because issuing stock at a low price during a downturn would be costly for current shareholders (and all the more costly that the company partly signals to the markets that it is not undervalued enough not to issue anymore by issuing in the downturn). Holding cash will thus have the advantage of avoiding this situation. It will thus be all the more pertinent that companies are subject to this type of overvaluation and undervaluation issues, a fact more frequent for companies with high asymmetries of information (i.e. difficult to analyze fully by the markets, such as high technology firms, pharmaceuticals, and in general companies that have high R&D or high marketing expenses).

The value of X at time t can be expressed mathematically as

$$X_t = X_0 \cdot e^{-at} + b(1 - e^{-at}) + \sigma_x e^{-at} \int_0^t e^{as} dw_s$$

The value of being able to invest at time T if there is no underpricing is, as described in the previous section, the value of a European option that can be valued with the Black and Scholes formula. On the other hand, with the undervaluation occurring stochastically at time of exercise, then the payoff at time T of the option when there is the possibility of underpricing is given by

$$\max(V_T - K - X_T, 0)$$

We make all the usual assumption necessary technical assumptions for risk-neutral valuation and the standard real options framework. We can thus show that the value of the project taking into account the risk of underpricing at time T is given by the following expression (as proved in Appendix):

$$Option_{underpricing} = V_0 e^{-dT} A_1 - Ke^{-rT} A_2 - e^{-rT} A_3$$

$$A_1 = \int_{-\infty}^{\infty} f(v) N\left(-\frac{\phi(v+\rho\sigma_v\sqrt{T}) - \rho v - \sigma_v\sqrt{T}}{\sqrt{1-\rho^2}}\right) dv$$

$$A_2 = \int_{-\infty}^{\infty} f(v) N\left(-\frac{\phi(v) - \rho v}{\sqrt{1-\rho^2}}\right) dv$$

$$A_3 = \int_{-\infty}^{\infty} \left(X_\mu + \sqrt{X_{\sigma^2}} \cdot v\right) f(v) N\left(-\frac{\phi(v) - \rho v}{\sqrt{1-\rho^2}}\right) dv$$

$$X_\mu = X_0 \cdot e^{-aT} + b(1 - e^{-aT})$$

$$X_{\sigma^2} = \frac{\sigma_x^2}{2a}(1 - e^{-aT})$$

$$\phi(v) = \left(\ln\left(\frac{K + X_T(v)}{V_0}\right) - \left(r - d - \frac{1}{2}\sigma^2\right)T\right) \cdot \left(\sigma\sqrt{T}\right)^{-1}$$

where $f(\cdot)$ is the probability density function of a standard normal variable and $N(\cdot)$ is the cumulative density function of a standard normal variable. The values of the option to invest with the risk of under and overpricing are obtained by solving the above integral numerically. The value of the benefits of avoiding the underpricing issue is thus given by:

"Underpricing avoidance" benefit from holding cash

= European option (no underpricing risk)-European option (with underpricing risk)

The following graphs show how the benefit from avoiding the underpricing problem depends on the mean level of underpricing, the speed of reversion of that level, the volatility of the project returns as well as on the loss from waiting.

Figure 2a (overleaf) shows that, as expected, the benefits from avoiding underpricing rises as the mean level of underpricing rises. Also, the speed of mean reversion, i.e. the inverse of the time required to come back to the mean level when underpricing departs from that mean, affects the benefits of holding cash. High speeds of mean reversion lead to high benefits to avoiding the underpricing problem.

The figure overleaf shows the benefit from avoiding underpricing for different mean levels of underpricing and different speeds of adjustment. The values of the fixed parameters are: $V_0 = 100$; K= 90; $\sigma = 0.2$; a = 0.2, b = 3, $\sigma_x = 0.5$, $\rho = -0.2$, r = 0.08, loss rate = 0.12; T = 0.25.

Figure 2a Underpricing avoidance benefit

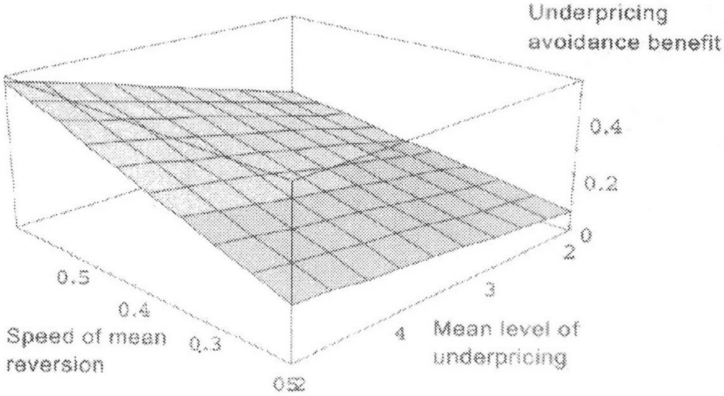
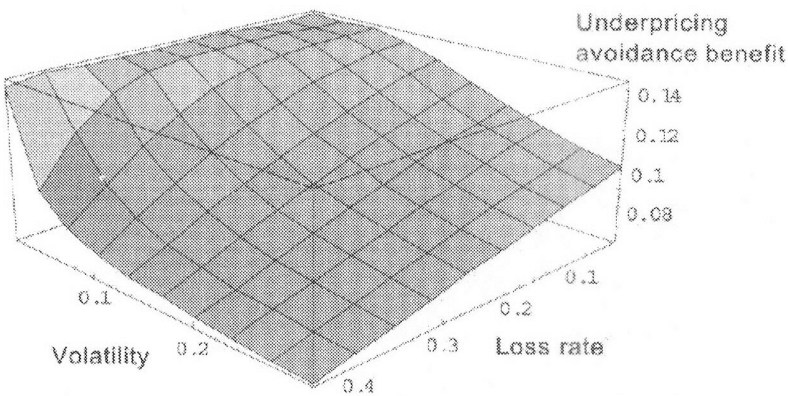


Figure 2 b goes in the direction expected from the previous section. Higher volatility leads to lower benefits at avoiding the underpricing problem. It is nonetheless interesting to notice that a higher loss at waiting leads to a lower benefit from avoiding the underpricing. Also this impact seems much smaller for reasonable values of the parameters than the inverse effect to the timing benefit, it shows once more that the use of the option pricing methodology leads to non trivial results.

Figure 2b Underpricing avoidance benefit



The above figure shows the benefit from avoiding underpricing for different volatilities of the project returns and different rates of loss at waiting from the project. The values of the fixed parameters are: $V_0 = 100$; $K= 90$; $X_0 = 0$; $a = 0.2$, $b = 3$, $\sigma_X = 0.5$, $\rho = -0.2$, $r = 0.08$, $T = 0.25$.

1.4 Combining the timing value with the underpricing avoidance value

Combining the two preceding parts and calculating the value of holding the necessary cash compared to raising the capital from outside sources and facing both the delay and the undervaluation risk and their interaction would be the natural next step. In particular, in our model, the advantage of holding cash is twofold. It provides the flexibility to go ahead with the project at any time before maturity and it eliminates the risk of having to raise equity or other types of financing when the security prices are undervalued. If the company holds enough cash to exercise the project at any time the value of the project corresponds to an American option. If the firm has to wait until it receives the new capital the value of the project is given by the European stochastic strike option. The overall benefits from holding cash can thus be written as

Cash holdings benefit = American option -European option (with underpricing risk)

Overall benefits from cash holdings can thus be calculated using the difference between a simple American option as can be obtained from numerical analysis and the specific European option with stochastic exercise price that we have derived above.

It is thus easy to check that the major results obtained above remain. In particular, holdings of cash should increase with the loss at waiting (for reasonable parameters value, also an inverse relationship exists for the underpricing avoidance benefits) as well as with the underpricing of securities in the markets. Nonetheless, the analysis provides for counterintuitive results as well, notably that the risk of the project returns and the risk of the undervaluation as well should lead to relatively smaller cash holdings. These counterintuitive results follow from the option form of the investment project considered, and notably from the fact that in high risk environments, there may be some value to wait before investing (and this value rises with uncertainty, as classically obtained with options).

1.5 Conclusion

We have derived here a real options model of the benefits of cash holdings. Cash holdings allow for optimal timing of an investment (the classical timing option of investments in the real options literature) while also avoiding the underpricing issue (as described in the information asymmetries literature). We combine the two phenomena in a homogenous framework that allow to value the overall benefits of cash holdings. Expected results follow, such as the impact of the loss at waiting and the impact of the underpricing: in both cases, more of any should lead corporates to hold more cash (at least for reasonable parameter values), and we are able to tell how much more cash (rather than be limited to qualitative results like some of the literature has been limited to in the past). Some unexpected results follow as well, unexpected from a cash management point of view but classical results in the option literature nonetheless: higher risk may lead to less need for cash as later exercise may become preferable or as underpricing may become favorable in the future. Also, some detailed impact could warrant further investigation in specific cases: for example, in the case of the underpricing avoidance option of cash holdings, a higher loss at waiting may lead to lower optimal cash holdings (via an indirect effect of the risk of a stochastic exercise price).

We believe that this paper lays the foundation for further theoretical and empirical research in the area as well as for practical applications of the stylized model proposed here. Obviously though, many improvements on the model have to be added, notably by making the overall set-up more realistic, with for example the addition of the stochasticity of the investment itself (a risk dimension that is not well accounted for here and that would probably lead to higher cash holdings) as well as combining benefits of cash with costs of cash in order to obtain an optimal level of cash.

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2 Appendix

This appendix shows the derivation of the valuation formula in section 2.

The value of the project with underpricing risk gain from default at time T is given by

$$\text{Project} = \max\,(V_T - K - X_T,\ 0) \tag{1}$$

We assume that the value of the project follows a geometric Brownian motion process. Hence it satisfies the following PDEs:

$$\frac{dV}{V} = (r-d)dt + \sigma du \tag{2}$$

Hence the value of the project at time t<T is given as

$$V(t) = S_t e^{(r-d-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}\,u} \tag{3}$$

The amount of underpricing is assumed to follow a mean reverting process given by

$$dX = a(b - X_t)dt + \sigma_x dz \tag{4}$$

Its value at time t can be obtained as

$$X_t = X_0 \cdot e^{-aT} + b\left(1 - e^{-at}\right) + \sigma_x e^{-aT}\int_0^t e^{as}dw_s \tag{5}$$

using then expected value of X_t and its variance this can be reexpressed as:

$$X_t = X_\mu + \sqrt{X_{\sigma^2}} \cdot v \tag{6}$$

where

$$X_\mu = X_0 \cdot e^{-at} + b(1 - e^{-at})$$
$$X_{\sigma^2} = \frac{\sigma_x}{2a}(1 - e^{-at})$$

The two Wiener processes are assumed to be dependent hence

$$du \cdot dv = \rho dt \tag{7}$$

The joint density function of two correlated standard normal variables is given as

$$f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2}\left(\frac{u^2-2\rho uv+v^2}{\sqrt{1-\rho^2}}\right)^2} \tag{8}$$

It will be seen later that it is more practical to evaluate the above expectation by using the following decomposition

$$f(u,v) = f(v) \cdot f(u \mid v) \tag{9}$$

$$f(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}$$

$$f(u \mid v) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)^2}$$

The second piece of information that is needed in order to evaluate the above expectation is the range of integration for the two random variables u and v.

The project will be worth zero if

$$V_T - K - X_T > 0 \tag{10}$$

hence we obtain

$$u > \left(\ln\left(\frac{K + X_T(v)}{V_0}\right) - \left(r - d - \frac{1}{2}\sigma^2\right)T \right) \cdot \left(\sigma\sqrt{T}\right)^{-1} = \phi(v) \tag{11}$$

Remember that X_t was defined above and it's a function of v. The range for v is from $-\infty$ to ∞.

In order to find the value of the project at time t=0 we need to solve the following integral

$$\text{Project} = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} (V_T - K - X_T) f(v) \cdot f(u \mid v) du\, dv \tag{12}$$

We will split the integral in three parts and solve them separately.

The three parts are the following ones:

$$Part\ I = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} V_T f(v) \cdot f(u \mid v) du\, dv \tag{13}$$

$$Part\ II = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} K f(v) \cdot f(u \mid v) du\, dv \tag{14}$$

$$Part\ III = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} X_T f(v) \cdot f(u \mid v) du\, dv \tag{15}$$

Integration of part I:

$$Part\ I = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} V_0 e^{\left(r-d-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\,u} \cdot f(v) \cdot f(u\,|\,v)\,du\,dv \tag{16}$$

$$Part\ I = V_0 e^{-dT} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2 T} f(v) \int_{\phi(v)}^{\infty} e^{\sigma\sqrt{T}\,u} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)^2} dv \tag{17}$$

In a first step we will simplify the terms in the second integral

$$\int_{\phi(v)}^{\infty} e^{\sigma\sqrt{T}u} \cdot \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)^2} dv \tag{18}$$

$$\int_{\phi(v)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u^2}{(1-\rho^2)} - \frac{2u}{(1-\rho^2)}(\rho v+\sigma\sqrt{T}(1-\rho^2)) + \frac{\rho^2 v^2}{(1-\rho^2)}\right)} dv \tag{19}$$

Completing the square in the exponent we get:

$$Part\ I = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} \tag{20}$$

$$\int_{\phi(v)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\left(\frac{u}{\sqrt{1-\rho^2}} - \left(\frac{\rho v}{\sqrt{1-\rho^2}} + \sigma\sqrt{T}\sqrt{1-\rho^2}\right)\right)^2 - 2\rho v\sigma\sqrt{T} - \sigma^2 T(1-\rho^2)\right)} dv\,du$$

$$Part\ I = V_o e^{-dT} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v^2-\sigma\sqrt{T}\rho)^2} \int_{\phi(v)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v-\sigma\sqrt{T}(1-\rho^2)}{\sqrt{1-\rho^2}}\right)^2} dv\,du \tag{21}$$

Rename $\bar{v} = v - \rho\sigma\sqrt{T}$, hence replace all v with $\bar{v} + \rho\sigma\sqrt{T}$ which yields:

$$Part\ I = V_0 e^{-dT} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\bar{v})^2} \int_{\phi(\bar{v}+\rho\sigma\sqrt{T})}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho\bar{v}-\sigma\sqrt{T}}{\sqrt{1-\rho^2}}\right)^2} d\bar{v}\,du \tag{22}$$

Set $X = \frac{u-\rho\bar{v}-\sigma\sqrt{T}}{\sqrt{1-\rho^2}}$ hence du=dx $\cdot\sqrt{1-\rho^2}$ and the lower limit of the integral becomes

$$x_{lower} = \frac{\phi\left(\bar{v} + \rho\sigma\sqrt{T}\right) - \rho\bar{v} - \sigma\sqrt{T}}{\sqrt{1-\rho^2}} \tag{23}$$

$$Part\ I = V_0 e^{-dT} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(\bar{v})^2} N(-x_{lower}) dv = V_0 e^{-dT} \cdot A_1$$

Rename $\bar{v} = v$.

Integration of part II:

$$Part\ II = e^{-rT} K \int_{-\infty}^{+\infty} f(v) \int_{\phi(v)}^{\infty} f(u \mid v) du\, dv \tag{24}$$

$$Part\ II = e^{-rT} K \int_{-\infty}^{+\infty} f(v) \int_{\phi(v)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)^2} du\, dv \tag{25}$$

$$Part\ II = e^{-rT} K \int_{-\infty}^{+\infty} f(v) N\left(-\frac{\phi(v) - \rho v}{\sqrt{1-\rho^2}}\right) dv = e^{-rT} K \cdot A_2 \tag{26}$$

Integration of part III:

$$Part\ III = e^{-rT} \int_{-\infty}^{+\infty} \int_{\phi(v)}^{\infty} X_T f(v) f(u \mid v) du\, dv \tag{27}$$

$$Part\ III = e^{-rT} \int_{-\infty}^{+\infty} \left(X_\mu + \sqrt{X_{\sigma^2}}\,.v\right) f(v) \int_{\phi(v)}^{\infty} f(u \mid v) du\, dv \tag{28}$$

$$Part\ III = e^{-rT} \int_{-\infty}^{+\infty} \left(X_\mu + \sqrt{X_{\sigma^2}} \cdot v\right) f(v) \int_{\phi(v)}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{u-\rho v}{\sqrt{1-\rho^2}}\right)^2} du\, dv \tag{29}$$

$$Part\ III = e^{-rT} \int_{-\infty}^{+\infty} \left(X_\mu + \sqrt{X_{\sigma^2}} \cdot v\right) f(v) N\left(-\frac{\phi(v) - \rho v}{\sqrt{1-\rho^2}}\right) dv = e^{-rT} \cdot A_3 \tag{30}$$

Taking all the parts together we obtain:

$$\text{Project value} = V_0 e^{-dT} \cdot A_1 - e^{-rT} K \cdot A_2 - e^{-rT} \cdot A_3 \tag{31}$$

where A_1, A_2 and A_3 are defined as above.